

Graph Connectivity and Universal Rigidity of Bar Frameworks *

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Abstract

Let G be a graph on n nodes. In this note, we prove that if G is $(r+1)$ -vertex connected, $1 \leq r \leq n-2$, then there exists a configuration p in general position in \mathbb{R}^r such that the bar framework (G, p) is universally rigid. The proof is constructive, and is based on a theorem by Lovász *et al* concerning orthogonal representations and connectivity of graphs [12, 13].

1 Introduction

Let $G = (V, E)$ be a graph on n nodes. G is said to be k -vertex connected, or simply k -connected, if $n = k + 1$ and G is the complete graph, or if $n \geq k + 2$ and there does not exist a set of $(k - 1)$ nodes whose deletion disconnects G . A *bar framework* in \mathbb{R}^r is a simple incomplete connected graph G whose nodes are points p^1, \dots, p^n in \mathbb{R}^r ; and whose edges are line segments, each joining a pair of these points. The points p^1, \dots, p^n will be denoted collectively by p , and the bar framework will be denoted by (G, p) . Also, we will refer to p as the *configuration* of the bar framework. A configuration p (or a framework (G, p)) is r -dimensional if the points p^1, \dots, p^n

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affinely span \mathbb{R}^r . Moreover, a configuration p (or a framework (G, p)) is in general position in \mathbb{R}^r if every $r + 1$ points in configuration p are affinely independent.

An r' -dimensional bar framework (G, p') is *equivalent* to an r -dimensional bar framework (G, p) if:

$$\|p'^i - p'^j\|^2 = \|p^i - p^j\|^2 \quad \text{for each } \{i, j\} \in E(G), \quad (1)$$

where $\|\cdot\|$ denotes the Euclidean norm and $E(G)$ denotes the edge set of graph G . On the other hand, two r -dimensional bar frameworks (G, p) and (G, p') are *congruent* if:

$$\|p'^i - p'^j\|^2 = \|p^i - p^j\|^2 \quad \text{for all } i, j = 1, \dots, n. \quad (2)$$

An r -dimensional bar framework (G, p) is said to be *universally rigid* if there does not exist an r' -dimensional bar framework (G, p') , where r' is a positive integer $\leq n - 1$, such that (G, p') is equivalent but not congruent to (G, p) .

An immediate necessary condition for an r -dimensional bar framework (G, p) on n nodes ($r \leq n - 2$) in general position in \mathbb{R}^r to be universally rigid is that graph G should be $(r + 1)$ -connected [10]. For suppose that G is not $(r + 1)$ -connected. Then there exists a set of r nodes, say X , whose removal disconnects G . Let $V(G) = V_1 \cup X \cup V_2$ be a partition of the nodes of G , where V_1 and V_2 are non-empty, such that there are no edges joining nodes in V_1 to nodes in V_2 . The points $\{p^i : i \in X\}$ lie in a hyperplane H in \mathbb{R}^r , and the points $\{p^i : i \in V_1 \cup V_2\}$ do not lie in H since p is in general position in \mathbb{R}^r . For all nodes $i \in V_2$, let q^i be the reflection of p^i with respect to H and let $p' = \{p^i : i \in (V_1 \cup X)\} \cup \{q^i : i \in V_2\}$. Thus (G, p') is an r -dimensional bar framework that is equivalent but not congruent to (G, p) , and hence (G, p) is not universally rigid. This raises the question of whether the assumption of $(r + 1)$ -connectivity of graph G alone is sufficient for the existence of some r -dimensional configuration p in general position in \mathbb{R}^r such that the bar framework (G, p) is universally rigid. The following theorem, which is our main result, is an affirmative answer to this question.

Theorem 1.1. *Let G be a graph on n nodes and assume that G is $(r + 1)$ -vertex connected, where $1 \leq r \leq n - 2$. Then there exists an r -dimensional bar framework (G, p) in general position in \mathbb{R}^r such that (G, p) is universally rigid.*

The proof of Theorem 1.1, which is given in Section 3, is constructive and is based on a theorem by Lovász *et al* [12, 13] concerning orthogonal representations and connectivity of graphs.

Note that the complete bipartite graph $K_{3,3}$ is 3-connected. Thus, Theorem 1.1 provides a negative answer to a question raised by Yinyu Ye as to whether in every universally rigid 2-dimensional bar framework (G, p) in general position in \mathbb{R}^2 , graph G must contain a triangle.

2 Preliminaries

This section presents the necessary mathematical background. The first subsection reviews basic definitions and results on stress and Gale matrices, and their role in the problem of universal rigidity. The second subsection focuses on vertex connectivity and orthogonal representations of graphs.

2.1 Stress and Gale Matrices

Stress matrices play a key role in the study of universal rigidity. An *equilibrium stress* (or simply a *stress*) of a bar framework (G, p) is a real-valued function ω on $E(G)$ such that:

$$\sum_{j: \{i,j\} \in E(G)} \omega_{ij}(p^i - p^j) = \mathbf{0} \text{ for each } i = 1, \dots, n. \quad (3)$$

Here we use the bold zero “ $\mathbf{0}$ ” to denote the zero vector or the zero matrix of appropriate dimensions. Let $E(\overline{G})$ denote the edge set of graph \overline{G} , the complement graph of G . i.e.,

$$E(\overline{G}) = \{\{i, j\} : i \neq j, \{i, j\} \notin E(G)\},$$

and let $\omega = (\omega_{ij})$ be a stress of (G, p) . Then the $n \times n$ symmetric matrix Ω where

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & \text{if } \{i, j\} \in E(G), \\ 0 & \text{if } \{i, j\} \in E(\overline{G}), \\ \sum_{k: \{i,k\} \in E(G)} \omega_{ik} & \text{if } i = j, \end{cases} \quad (4)$$

is called the *stress matrix associated with* ω , or a *stress matrix* of (G, p) . Sufficient and necessary conditions, in terms of stress matrices, for universal rigidity of bar frameworks are discussed in [1, 6, 5, 2, 8]. The first sufficient condition for universal rigidity under the assumption that configuration p is in general position was given in [4].

Theorem 2.1 (Alfakih and Ye [4]). *Let (G, p) be an r -dimensional bar framework on n nodes in \mathbb{R}^r , for some $r \leq n - 2$. If the following two conditions hold:*

1. *There exists a positive semidefinite stress matrix Ω of (G, p) of rank $n - r - 1$,*
2. *The configuration p is in general position.*

Then (G, p) is universally rigid.

Theorem 2.1 was generalized and strengthened in [3], but it will suffice for the purposes of this note.

Stress matrices are intimately related to Gale matrices and Gale transform [7, 9]. This relation is a crucial step in connecting stress matrices to orthogonal representations of graphs. Given an r -dimensional bar framework (G, p) in \mathbb{R}^r , let

$$P := \begin{bmatrix} (p^1)^T \\ \vdots \\ (p^n)^T \end{bmatrix}. \quad (5)$$

Then P is called the *configuration matrix* of p (or of framework (G, p)). Moreover, let

$$Z := \begin{bmatrix} (z^1)^T \\ \vdots \\ (z^n)^T \end{bmatrix} \quad (6)$$

be any $n \times (n - r - 1)$ matrix whose columns form a basis of the null space of the matrix

$$\begin{bmatrix} P^T \\ e^T \end{bmatrix}, \quad (7)$$

where e is the vector of all 1's in \mathbb{R}^n . Note that the matrix in (7) has full row rank since (G, p) is r -dimensional. Then Z is called a *Gale matrix* of configuration p (or of bar framework (G, p)), and z^1, \dots, z^n are called, respectively, *Gale transforms* of p^1, \dots, p^n . Note that z^1, \dots, z^n are vectors in \mathbb{R}^{n-r-1} . Also, note that if Z is a Gale matrix of (G, p) and Q is any nonsingular matrix of order $n - r - 1$, then ZQ is also a Gale matrix of (G, p) .

It readily follows from (4) that a stress matrix Ω satisfies the following equations:

$$\Omega P = \mathbf{0}, \Omega e = \mathbf{0} \text{ and } \Omega_{ij} = 0 \text{ for each } \{i, j\} \in E(\overline{G}).$$

Thus, the columns of Ω belong to the null space of the matrix in (7). Accordingly, we have the following lemma.

Lemma 2.1 (Alfakih [2]). *Let (G, p) be an r -dimensional bar framework on n nodes in \mathbb{R}^r , $r \leq n - 2$, and let Z be a Gale matrix of (G, p) . Further, let $\Omega = Z\Psi Z^T$ for some symmetric matrix Ψ of order $n - r - 1$. If*

$$(Z\Psi Z^T)_{ij} = 0 \text{ for each } \{i, j\} \in E(\overline{G}), \quad (8)$$

then Ω is a stress matrix of (G, p) and $\text{rank } \Omega = \text{rank } \Psi \leq n - r - 1$.

A point worth observing here is that, by Lemma 2.1, Ω is positive semidefinite with rank $n - r - 1$ if and only if Ψ is positive definite if and only if there exist Gale transforms z'^1, \dots, z'^n of p^1, \dots, p^n such that $(z'^i)^T z'^j = 0$ for each $\{i, j\} \in E(\overline{G})$.

The usefulness of Gale transform in the study of universal rigidity under the general position assumption stems from the fact that Gale matrix Z , and consequently Gale transforms z^1, \dots, z^n , encode the affine dependence of points p^1, \dots, p^n .

Lemma 2.2. *Let (G, p) be an r -dimensional bar framework on n nodes in \mathbb{R}^r and let z^i be a Gale transform of p^i for $i = 1, \dots, n$. Then (G, p) is in general position if and only if any size- $(n - r - 1)$ subset of $\{z^1, \dots, z^n\}$ is linearly independent; i.e., any $(n - r - 1) \times (n - r - 1)$ submatrix of Gale matrix Z is nonsingular.*

For a proof of Lemma 2.2 see e.g. [1].

2.2 Graph Connectivity and Orthogonal Representations

An orthogonal representation of a graph G in \mathbb{R}^k is a mapping of each node i of G into a vector x^i in \mathbb{R}^k such that x^i is orthogonal to x^j for every pair of nonadjacent nodes i and j of G ; i.e., $(x^i)^T x^j = 0$ for each $\{i, j\} \in E(\overline{G})$. The vectors x^1, \dots, x^n are called the *representing vectors*. Orthogonal representations of graphs were introduced by Lovász in [11] in his study of the Shannon capacity of a graph. Obviously, $x^i = \mathbf{0}$ for each node i of G is a trivial orthogonal representation of G . Thus, in order to exclude such degenerate cases, orthogonal representations are required to satisfy the condition that any size- k subset of $\{x^1, \dots, x^n\}$ is linearly independent. The following theorem by Lovász *et al* is crucial for this note.

Theorem 2.2 (Lovász *et al* [12, 13]). *Let G be a graph on n nodes, then G is $(r + 1)$ -vertex connected, $r \leq n - 2$, if and only if G has an orthogonal representation in \mathbb{R}^{n-r-1} such that every size- $(n - r - 1)$ subset of the representing vectors is linearly independent.*

Let G be a graph on n nodes such that each node of G has a degree at least $r + 1$; i.e., each node of G has at most $n - r - 2$ non-adjacent nodes. Lovász *et al* [12, 13] presented the following simple randomized algorithm to construct an orthogonal representation of G in \mathbb{R}^{n-r-1} . Fix an ordering $(1, \dots, n)$ of the nodes of G . Then the representing vectors x^1, \dots, x^n are selected sequentially as follows. Select x^1 to be a uniformly random unit vector in \mathbb{R}^{n-r-1} . For $j = 2, \dots, n$, having selected x^1, \dots, x^{j-1} , select x^j to be a uniformly random unit vector from the subspace of \mathbb{R}^{n-r-1} that is orthogonal to the span of $\{x^i : i < j \text{ and } \{i, j\} \in E(\overline{G})\}$. This subspace has dimension ≥ 1 since the dimension of the span of $\{x^i : \{i, j\} \in E(\overline{G})\}$ is $\leq n - r - 2$. Now Lovász *et al* proved that if, in addition, G is $(r + 1)$ -connected, then, with probability 1, the orthogonal representation constructed by this algorithm has the property that every size- $(n - r - 1)$ subset of $\{x^1, \dots, x^n\}$ is linearly independent.

3 Proof of Theorem 1.1

Let G be a graph on n nodes and assume that G is $(r+1)$ -connected. Then by Theorem 2.2 there exist vectors x^1, \dots, x^n in \mathbb{R}^{n-r-1} such that $(x^i)^T x^j = 0$ for each $\{i, j\} \in E(\overline{G})$; and every size- $(n-r-1)$ subset of $\{x^1, \dots, x^n\}$ is linearly independent.

Let X^T be the $(n-r-1) \times n$ matrix whose i th column is equal to x^i ; i.e.,

$$X^T = \begin{bmatrix} x^1 & x^2 & \dots & x^n \end{bmatrix}. \quad (9)$$

Then $(XX^T)_{ij} = (x^i)^T x^j = 0$ for each $\{i, j\} \in E(\overline{G})$ and any square submatrix of X^T of order $n-r-1$ is nonsingular. The following two simple lemmas are needed.

Lemma 3.1. *Let X^T be the matrix defined in (9), then there exists a vector $\xi = (\xi_i)$ in \mathbb{R}^n such that $X^T \xi = \mathbf{0}$ and $\xi_i \neq 0$ for each $i = 1, \dots, n$.*

Proof. Without loss of generality let $\begin{bmatrix} I_{r+1} \\ B \end{bmatrix}$ be the $n \times (r+1)$ matrix whose columns form a basis of the null space of X^T , where I_{r+1} denotes the identity matrix of order $r+1$, and B is an $(n-r-1) \times (r+1)$ matrix. Then each column of $\begin{bmatrix} I_{r+1} \\ B \end{bmatrix}$ has exactly r zero entries; i.e., B has no zero entries. For suppose that B has a zero entry, say $B_{11} = 0$. Then the size- $(n-r-1)$ set $\{x^1, x^{r+3}, x^{r+4}, \dots, x^n\}$ is linearly dependent, a contradiction. Therefore, ξ is obtained by an appropriate linear combination of the columns of $\begin{bmatrix} I_{r+1} \\ B \end{bmatrix}$. □

Lemma 3.2. *Let X and ξ be as in Lemma 3.1 and let $Z = \text{Diag}(\xi)X$, where $\text{Diag}(\xi)$ is the diagonal matrix formed from the vector ξ . Furthermore, let P be the $n \times r$ matrix whose columns form a basis of the null space of*

$$\begin{bmatrix} Z^T \\ e^T \end{bmatrix}, \quad (10)$$

and let p be the configuration in \mathbb{R}^r whose configuration matrix is P . Then p is in general position in \mathbb{R}^r and Z is a Gale matrix of p .

Proof. Note that Z is $n \times (n-r-1)$ and $Z^T e = X^T \xi = \mathbf{0}$. Since ξ has no zero entries, the matrix $\text{Diag}(\xi)$ is nonsingular. Thus Z has full column rank and hence, by the definition of P in the lemma, it follows that Z is a Gale matrix of configuration p . Furthermore, every square submatrix of Z of order $n-r-1$ is

nonsingular. Therefore, by Lemma 2.2, configuration p is in general position in \mathbb{R}^r .

□

To complete the proof of Theorem 1.1, let $\Omega = ZZ^T = \text{Diag}(\xi)XX^T\text{Diag}(\xi)$. Then, obviously, Ω is positive semidefinite of rank $n - r - 1$. Moreover, let $\{i, j\} \in E(\overline{G})$, then $\Omega_{ij} = \xi_i\xi_j(XX^T)_{ij} = 0$. Hence, Ω is a stress matrix of the r -dimensional framework (G, p) in \mathbb{R}^r whose configuration matrix P is as given in Lemma 3.2. Since (G, p) is in general position, it follows from Theorem 2.1 that (G, p) is universally rigid.

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